Graph Reconstruction

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Abstract

The concept of a graph is one of the most basic and readily understood mathematical concepts, and the **Reconstruction Conjecture** is one of the most engaging problems under the domain of Graph Theory. The conjecture proposes that every graph with at least three vertices can be uniquely reconstructed given the multiset of subgraphs produced by deleting each vertex of the original graph one by one. This conjecture has been proven true for several infinite classes of graphs, but the general case remains unsolved. In this talk, I will outline the problem, and introduce some of the most well known results.

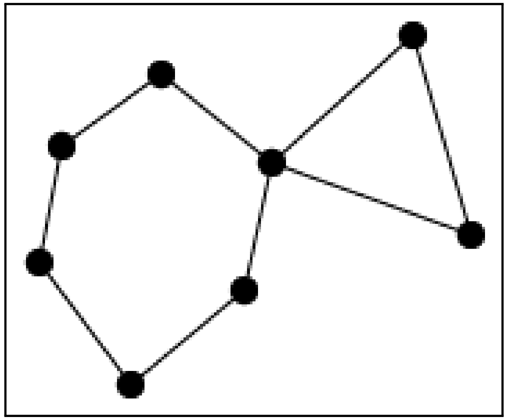
Graph Theory

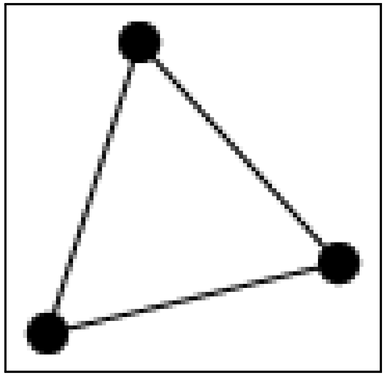
The study of Graphs ...

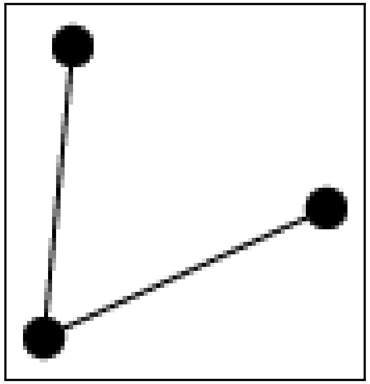
Graph Theory

The study of Graphs ... so what's a graph?

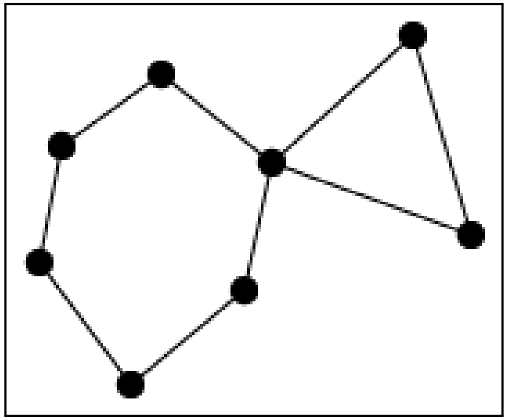
Graph Theory

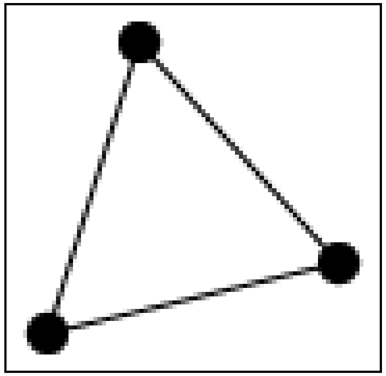


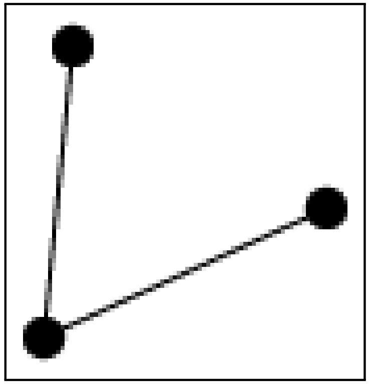




Graph Theory







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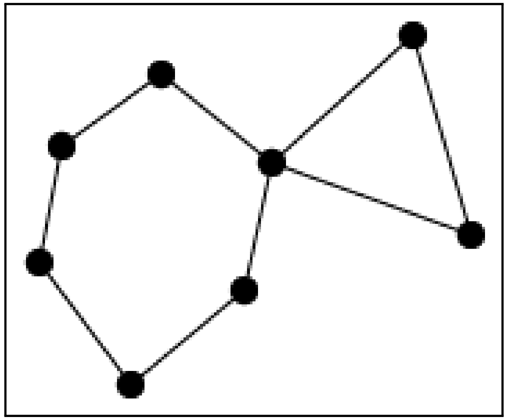

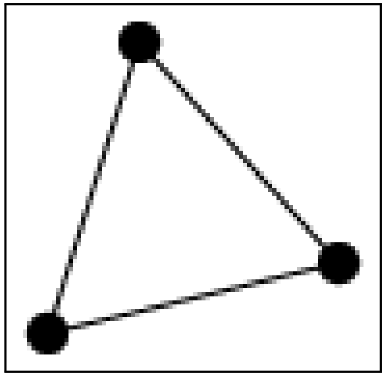

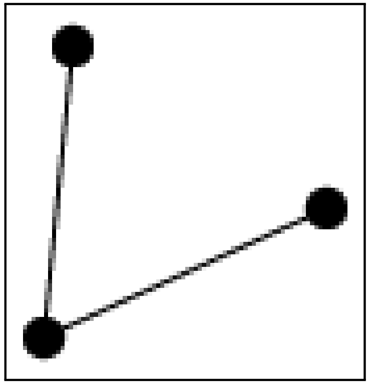

- $V(G)$ = the set of vertices of G.
- $E(G)$ = the set of edges of G.

Definitions

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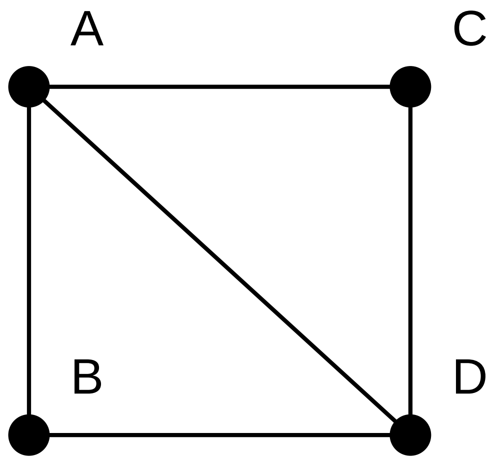
- A **card** of G is an unlabelled graph formed by deleting 1 vertex and all edges attached to it.
- The **deck** of G, $\mathscr{D}(G)$, is the collection of all G's cards. Note this is in general a multiset.

Example



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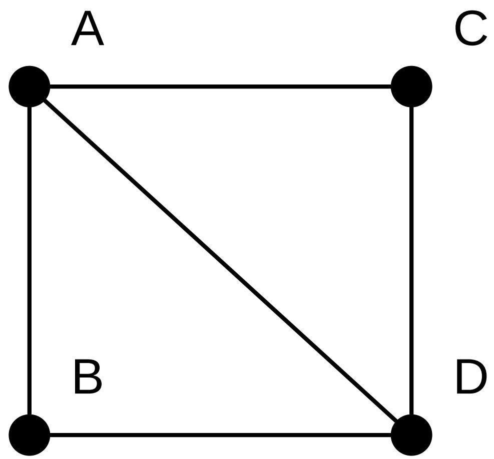

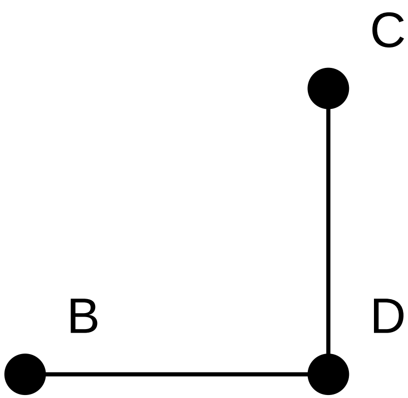

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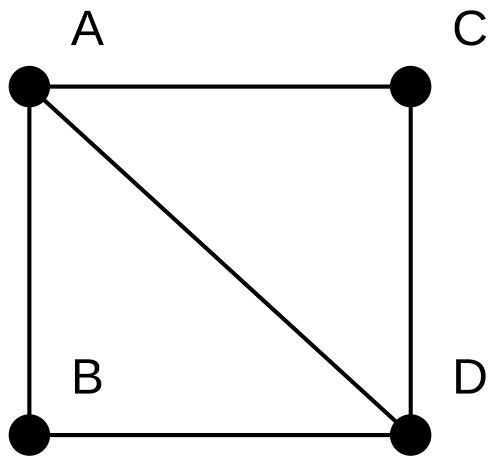

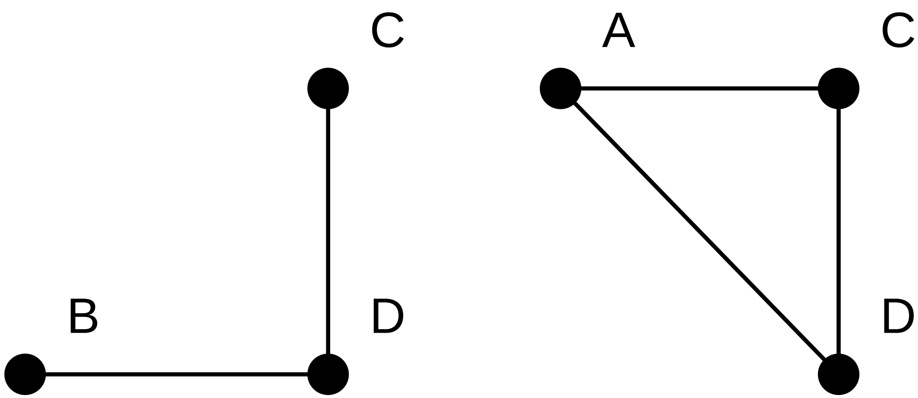

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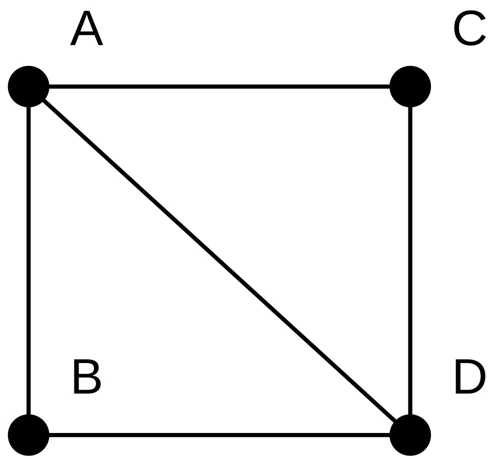

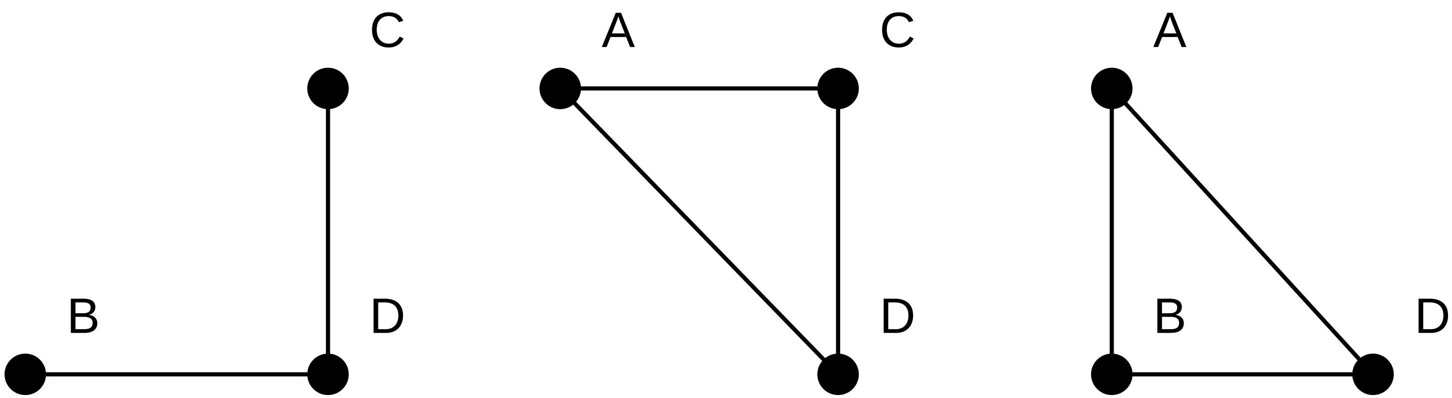

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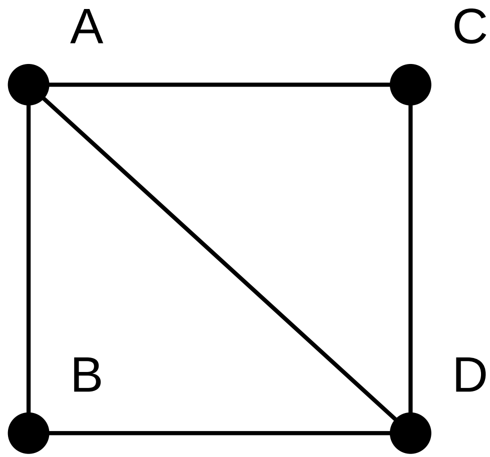

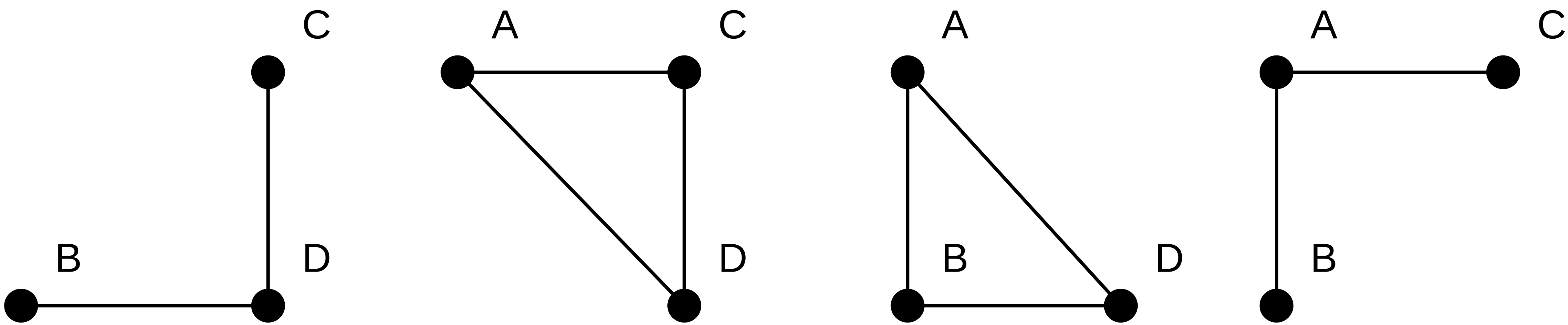

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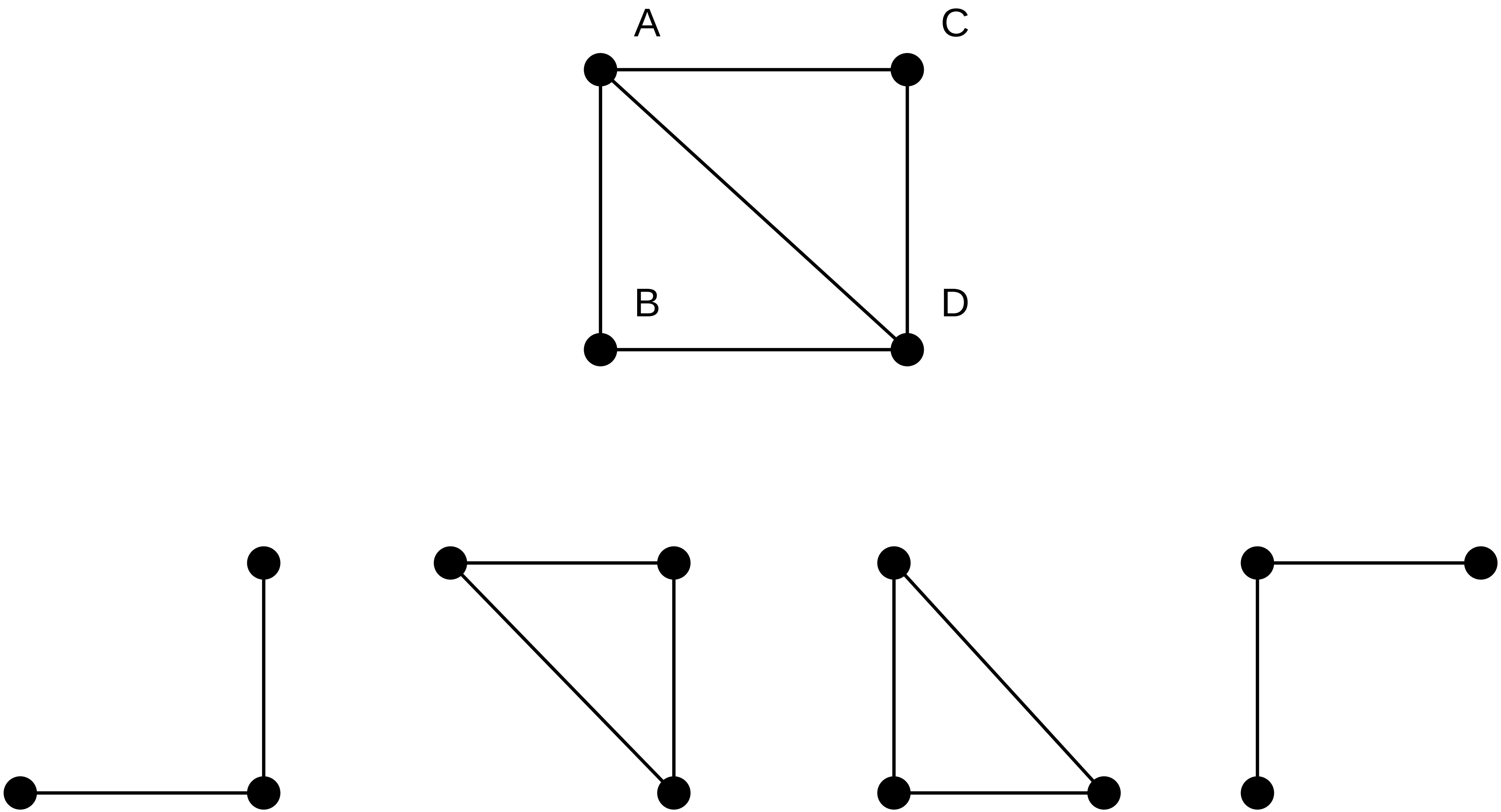

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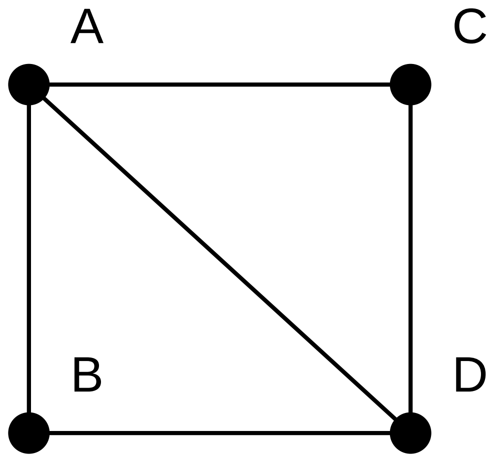

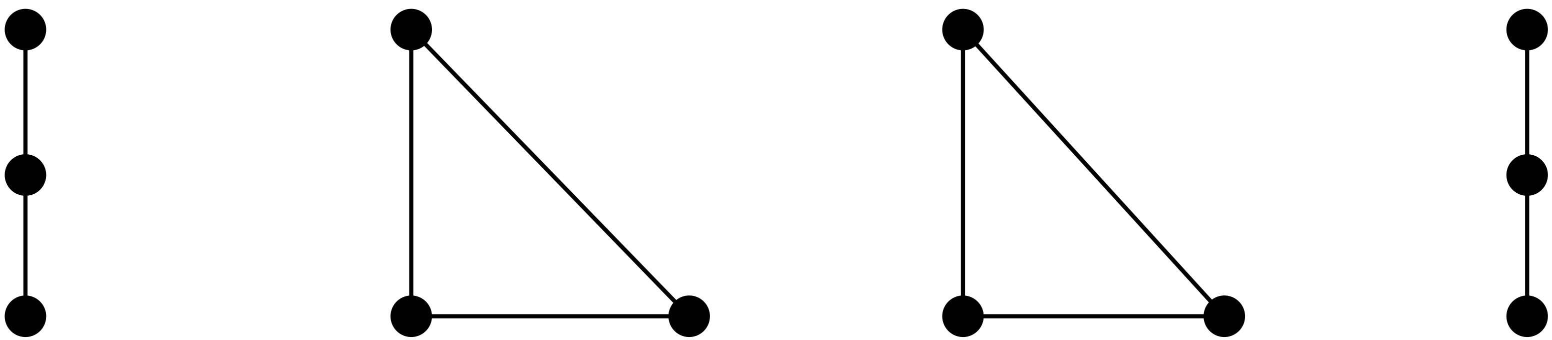

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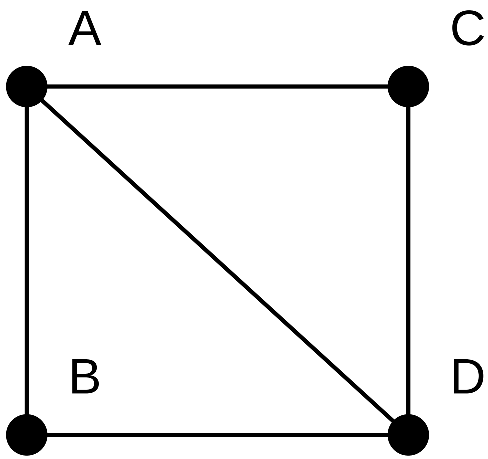

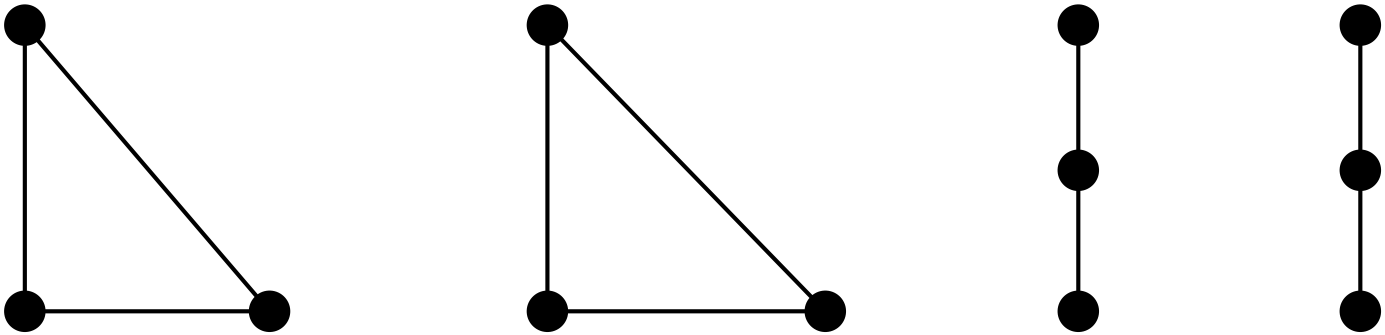

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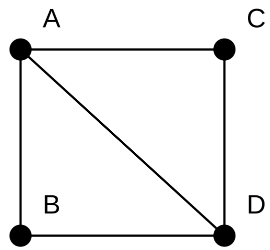

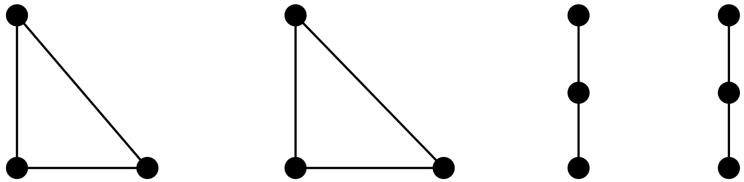

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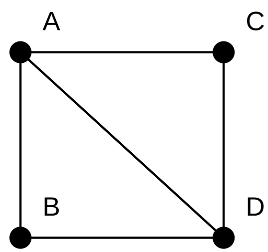

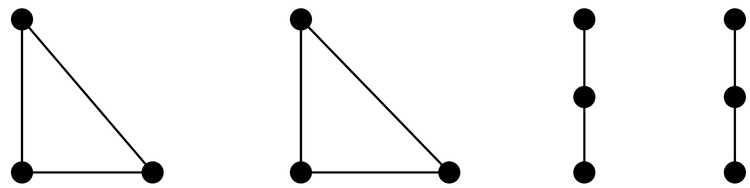

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- Equivalently, G is not reconstructible if and only if there is an H such that $\mathscr{D}(G) = \mathscr{D}(H)$.

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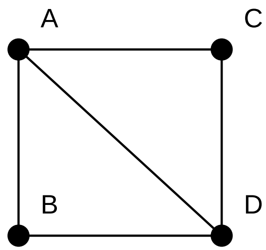
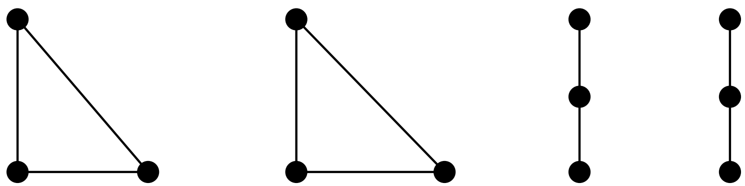


- This graph is reconstructible.



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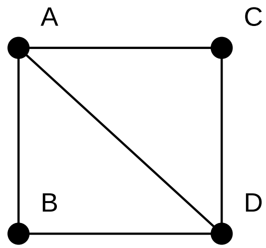
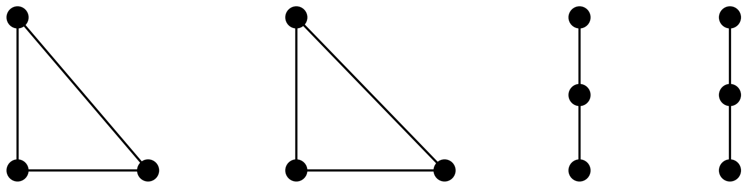
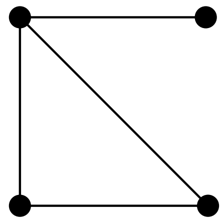




- This graph is reconstructible.
- The last three cards are also in the deck of



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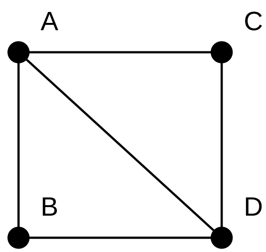

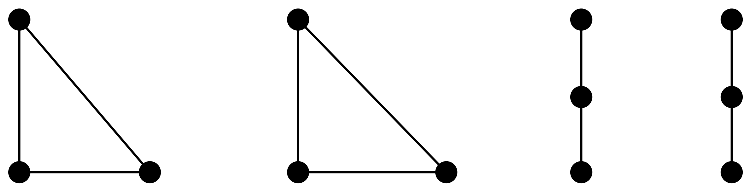

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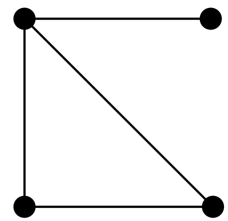

- (these can be checked by exhaustion).

The Reconstruction Conjecture

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- Note: the reconstruction conjecture is false if and only if there exist two graphs with the same deck.

Reconstructible Properties

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  - $|V(G)|$ = # of cards in $\mathscr{D}(G)$.
  - $|E(G)|$: Each edge in G is missing from exactly two cards. Thus each edge is in exactly $n-2$ cards, and we have

$$|E(G)| = \frac{1}{n-2} \sum_{v \in V(G)} |E(G-v)|.$$

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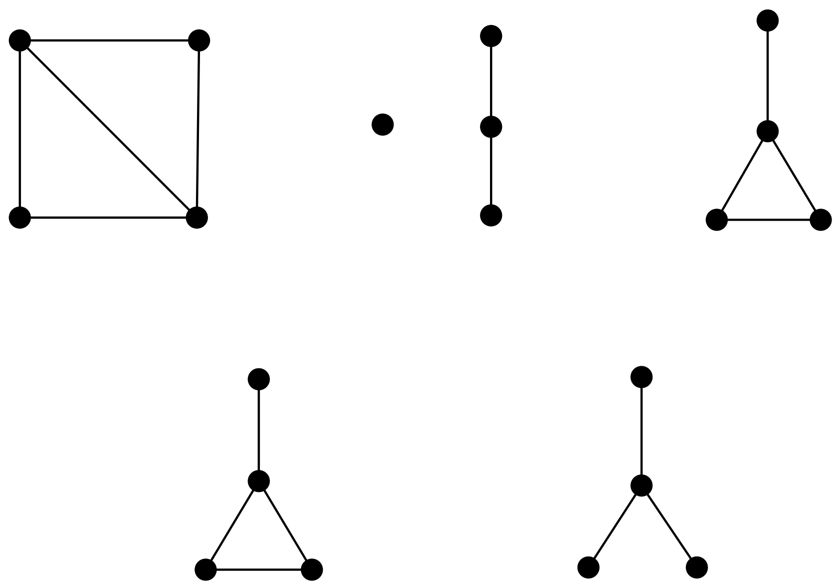
$$deg(v) = |E(G)| - |E(G - v)|.$$

- Since $|E(G)|$ can be determined from $\mathscr{D}(G)$, we can also determine the degree of v for all $v \in V(G)$.
- $\implies$ We can determine the whole *degree sequence* of G.

Reconstruction Example

Reconstruct, if possible, one or more graphs from the following deck:



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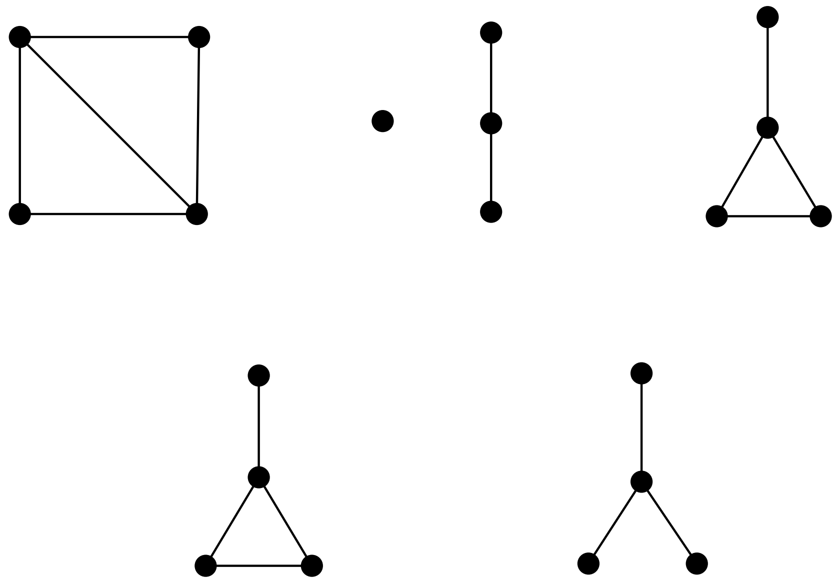

$|V(G)| =$ # of cards.

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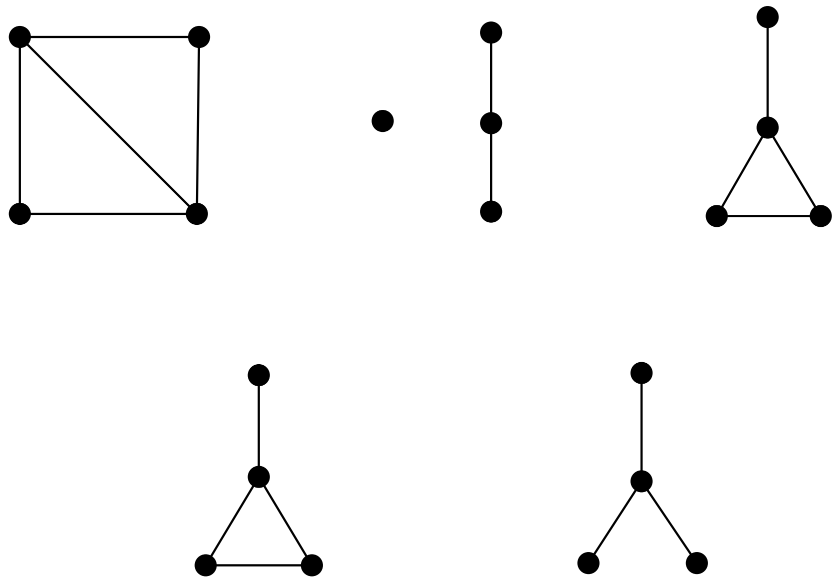

$$|V(G)| = \mathbf{5}.$$

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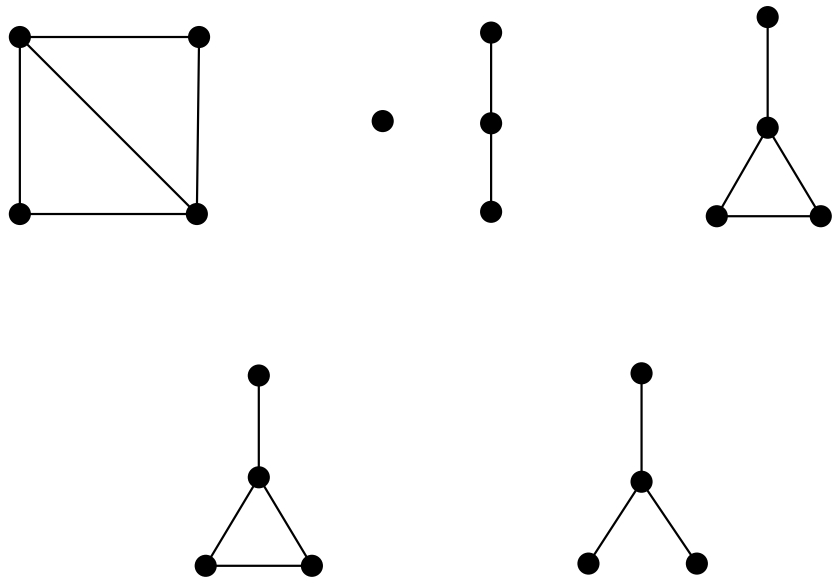

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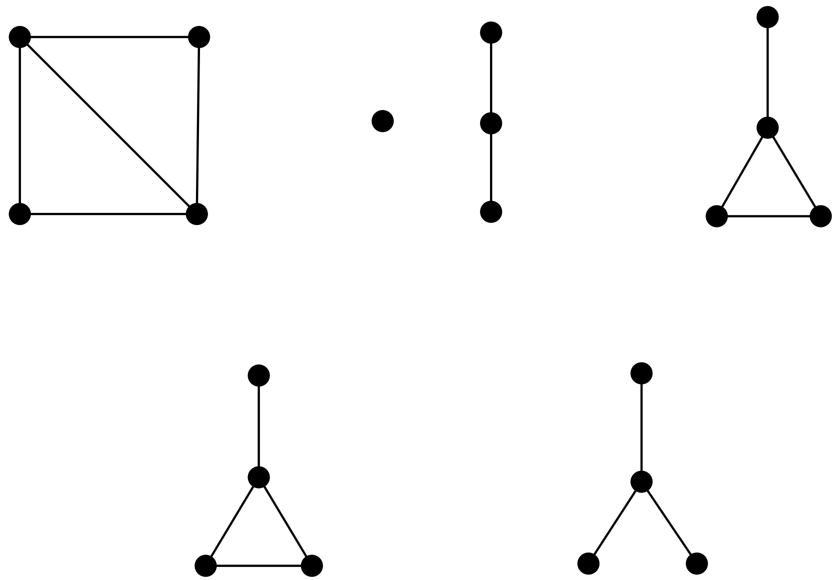

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$$|E(G)| = \frac{1}{|V(G)| - 2} \sum_{v \in V(G)} |E(G - v)|.$$

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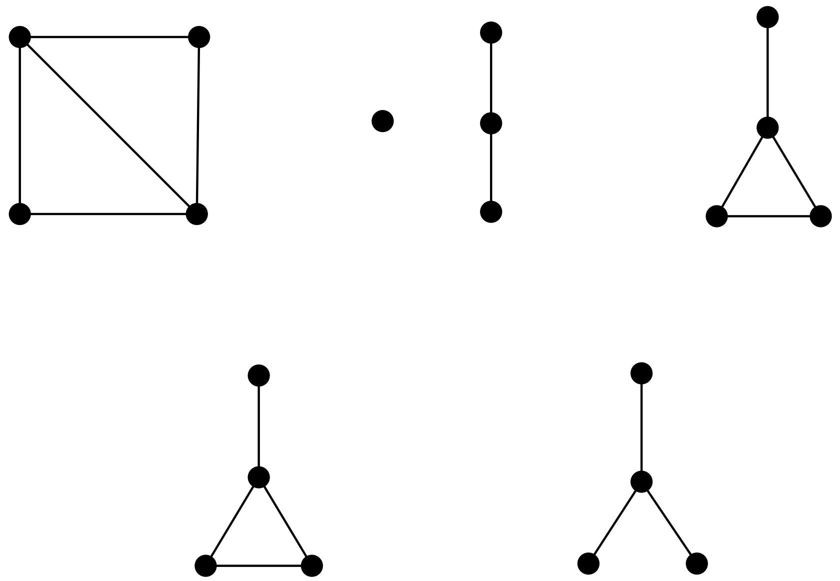

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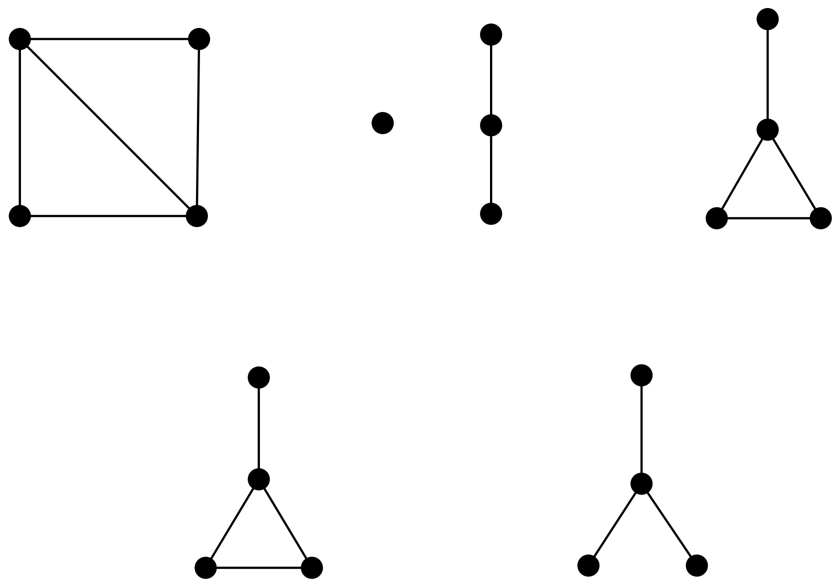

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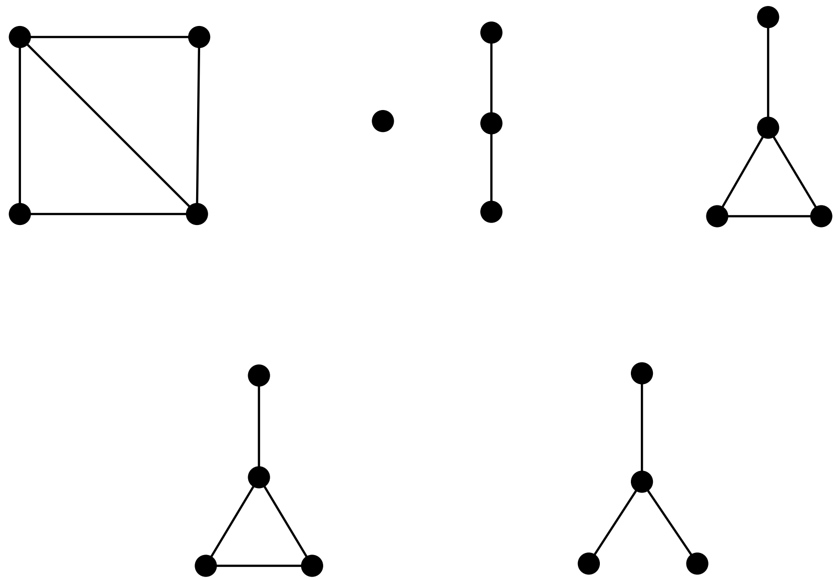

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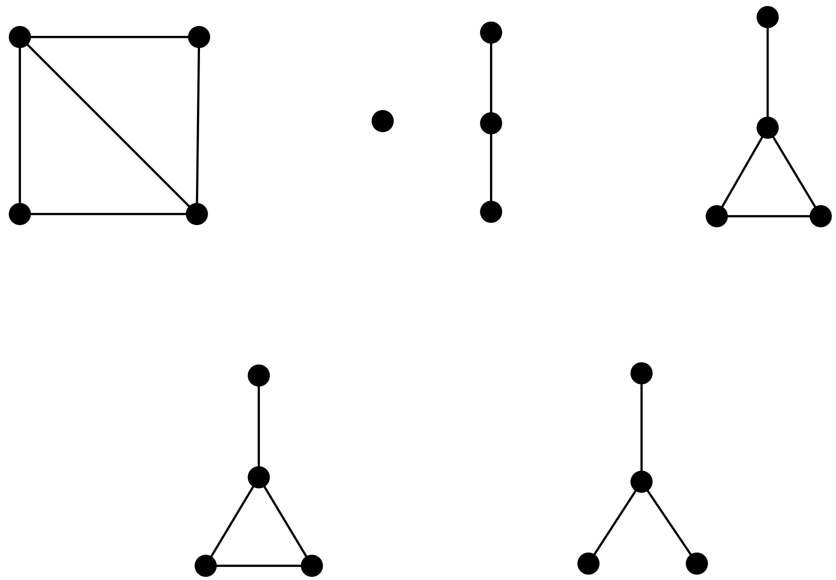

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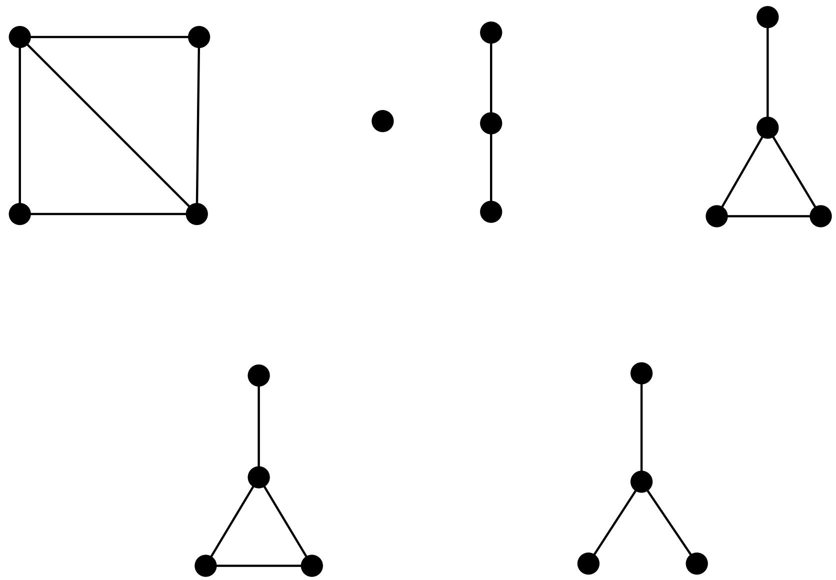

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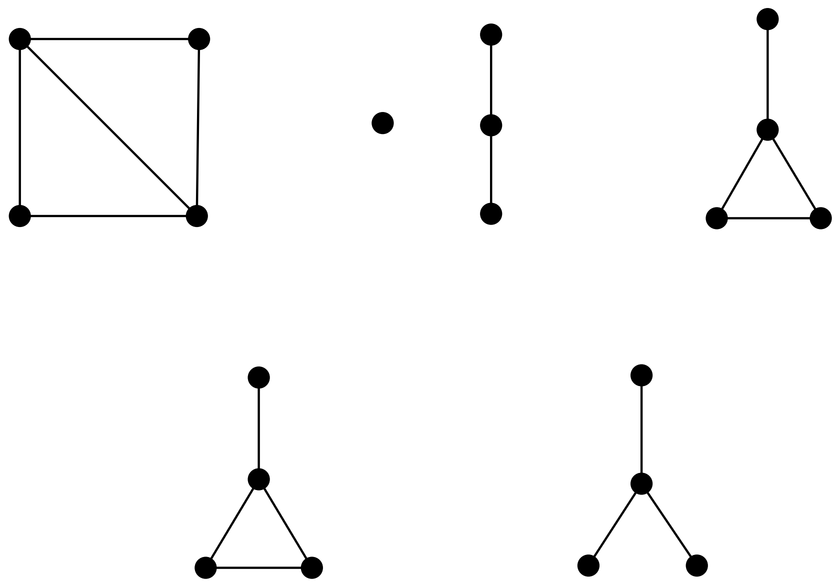

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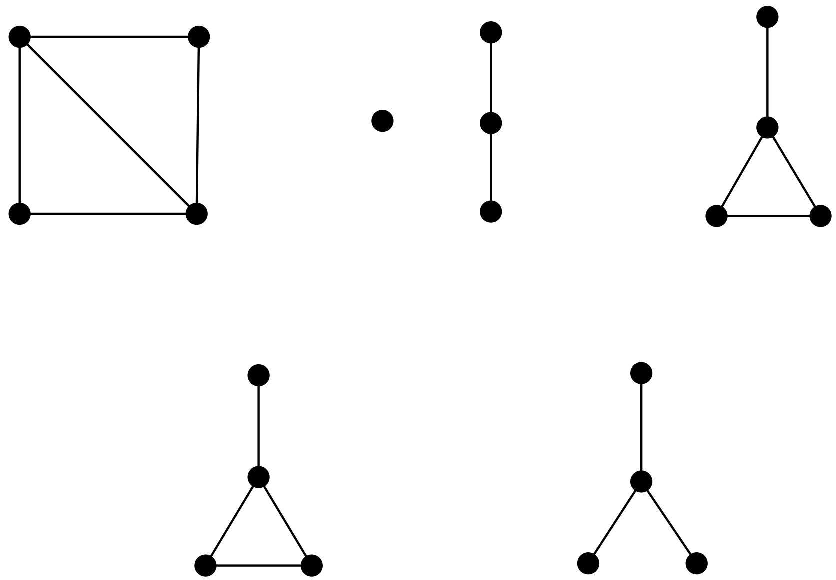

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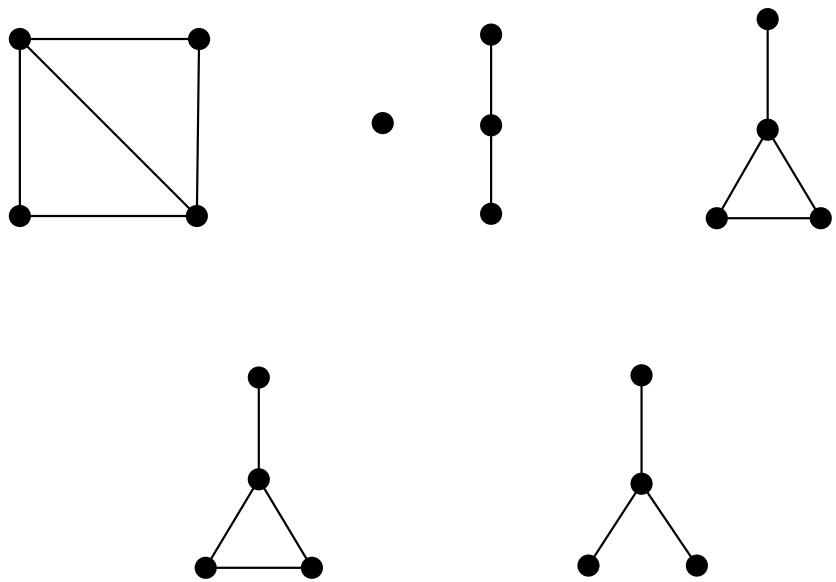

- $|V(G)| = 5$.
- $|E(G)| = 6$.
- Thus the degrees of the missing vertices in order are 1,4,2,2,3.

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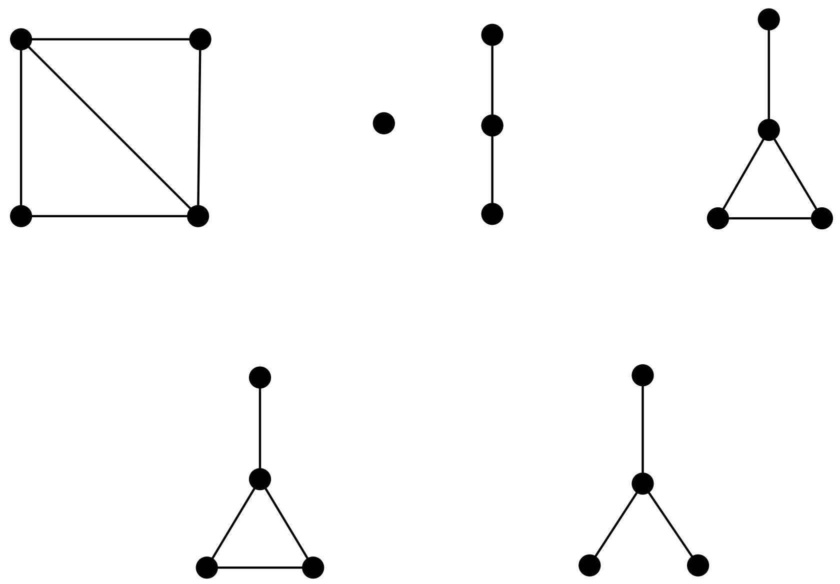

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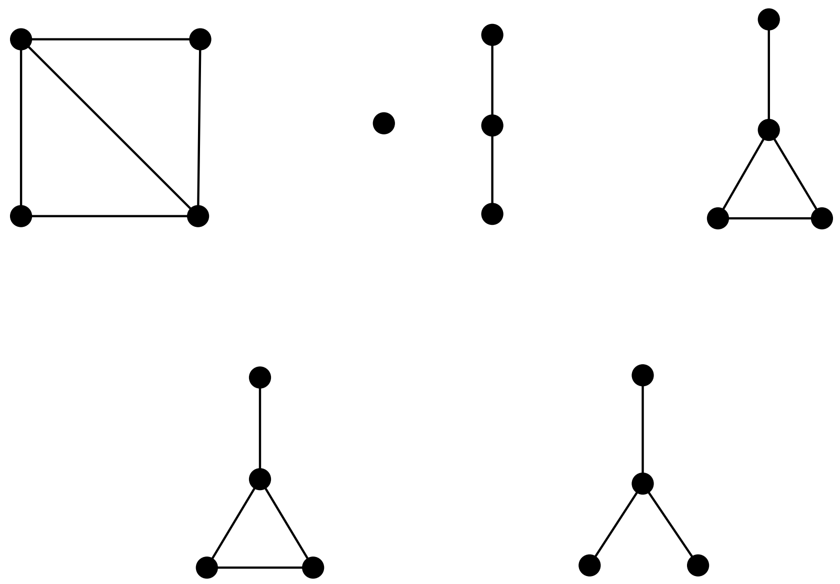

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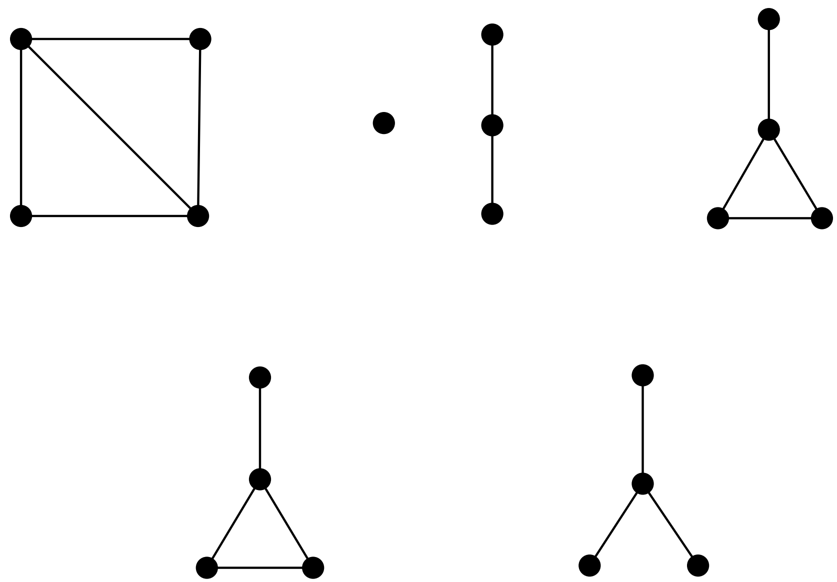

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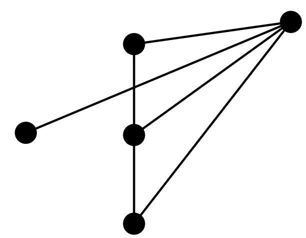

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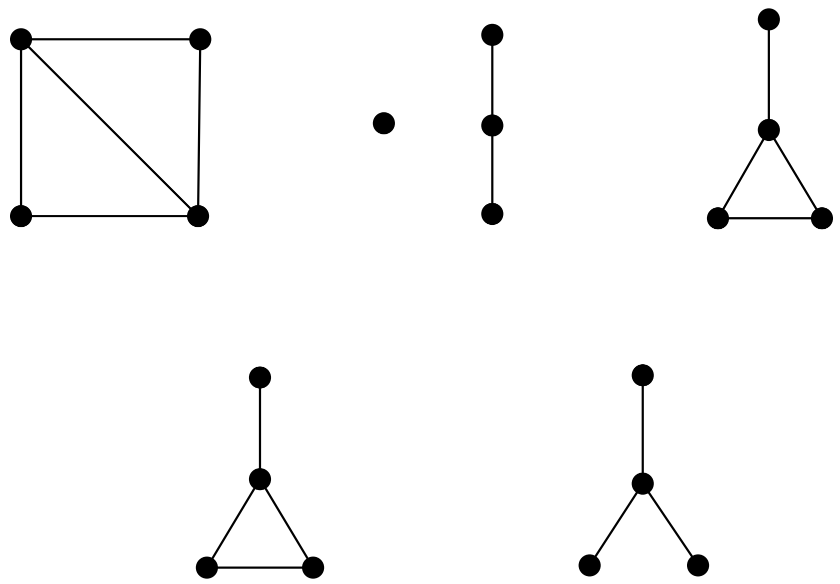

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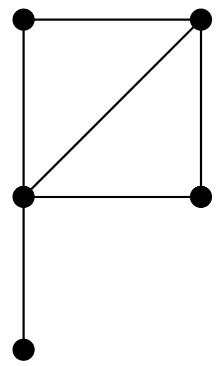

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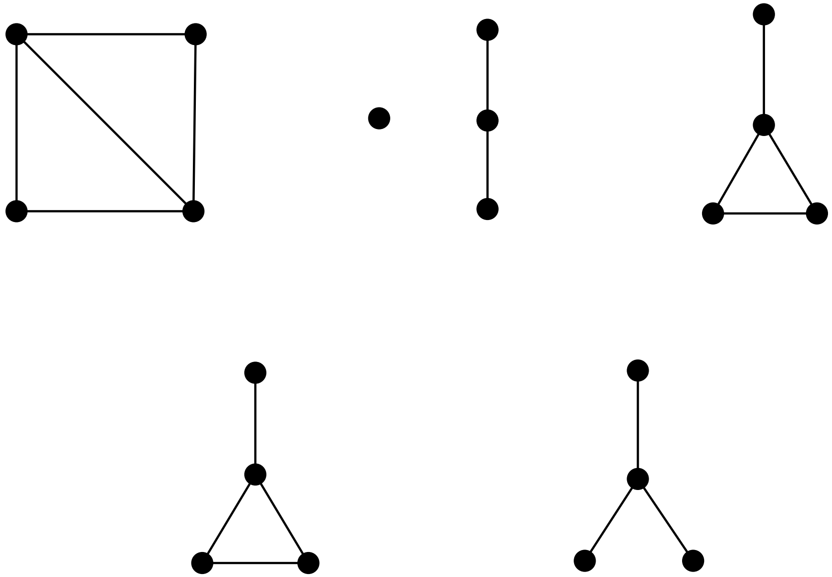

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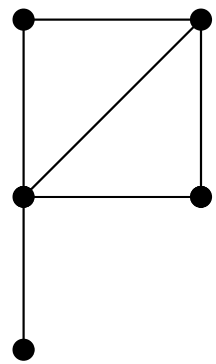

Determined Uniquely!

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- Connect new vertex to each of the d vertices in the card with degree $d - 1$, and G is reconstructed uniquely.

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- **Kelly's Theorem:** For all F with less vertices than G,

$$s(F, G) = \frac{1}{(|V(G)| - |V(F)|)} \sum_{v \in V(G)} s(F, G - v).$$

Proof of Kelly's Theorem:

Double count ordered pairs of the form

$$(H, G - v)$$

where $H \cong F$, and $H \subset G - v$.

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Thus, we have that Kelly's Lemma implies that the number of edges is determined by the deck.

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- Thus if there are two graphs with the same deck, they must have the same:
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  - number of edges.
  - number of subgraphs of EVERY type (same $s(F, G)$ for all F with $|V(F)| < |V(G)|$).

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- However, there are good reasons to believe it false...

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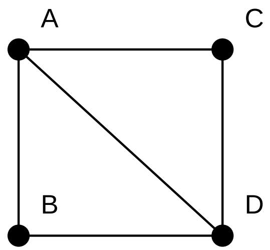
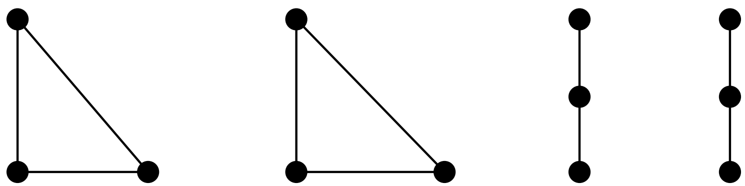
- The **universal reconstruction number** of G, $\forall rn(G)$, is the minimum number n such that all multisubsets of $\mathscr{D}(G)$ of size n uniquely reconstruct G.
- Almost all graphs have $\exists rn(G) = 3$. (Bollobas, 1990).

Example





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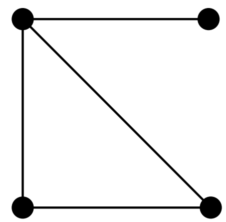.

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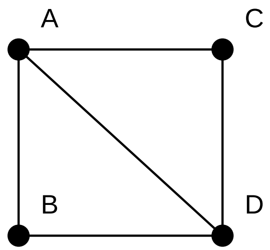

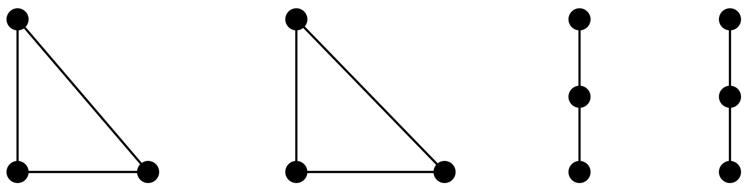

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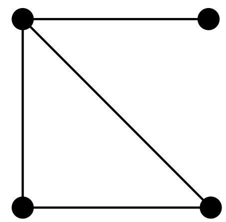

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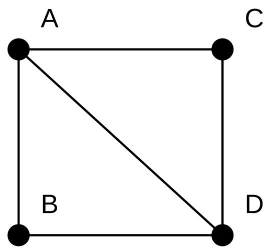

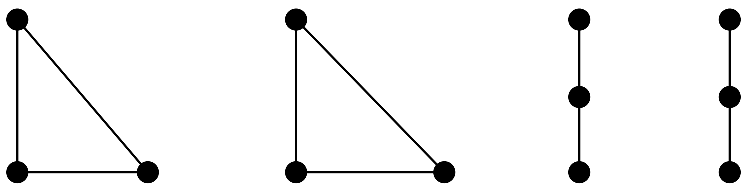

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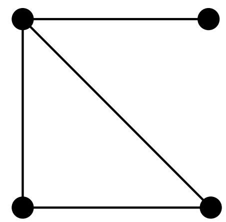.

- However, the first three graphs DO determine the graph G uniquely.
- Thus, for this G, $\exists rn(G) = 3, \forall rn(G) = 4$.

Reconstruction Numbers

- Found for all graphs on at most 10 vertices:

$\exists rn$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
3	4	8	34	150	1044	12,334	274,666	12,005,156
4		3		4		8		6
5				2		2	2	4
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- Very hard to get any further: there are 1,018,997,864 graphs on 11 vertices!

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  - The Graph Reconstruction Conjecture says $f(1) = 2$.

References

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